

IMSC 2058 Solution for Homework 4

Claim 1:

Let (x_n) and (y_n) be Cauchy sequences in X . Since

$$|d(x_n, y_n) - d(x_m, y_m)| \leq |d(x_n, x_m)| + |d(y_n, y_m)|$$

Then $(d(x_n, y_n))_{n=1}^{\infty}$ is a Cauchy sequence in \mathbb{R} . Note that \mathbb{R} is complete, therefore $\lim_n d(x_n, y_n)$ exists.

Let (x'_n) and (y'_n) be Cauchy sequences in X such that $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$, i.e. $\lim_n d(x_n, x'_n) = 0$ and $\lim_n d(y_n, y'_n) = 0$.

Since

$$d(x'_n, y'_n) \leq d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n).$$

It follows that

$$\tilde{d}([(x'_n)], [(y'_n)]) \leq \tilde{d}([(x_n)], [(y_n)])$$

Similarly, taking limit as $n \rightarrow \infty$ for the following inequality

$$d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n).$$

gives

$$\tilde{d}([(x_n)], [(y_n)]) \leq \tilde{d}([(x_n)], [(y_n)])$$

Therefore $\tilde{d}([(x'_n)], [(y'_n)]) = \tilde{d}([(x_n)], [(y_n)])$ when $(x_n) \sim (x'_n)$, $(y_n) \sim (y'_n)$.

Claim 2:

To show (\tilde{X}, \tilde{d}) is a metric, we verify :

- $\tilde{d}([(x_n)], [(y_n)]) = \lim_n d(x_n, y_n) \geq 0$.
- $\tilde{d}([(x_n)], [(x_n)]) = \lim_n d(x_n, x_n) = 0$.
- $\tilde{d}([(x_n)], [(y_n)]) = \lim_n d(x_n, y_n) = \lim_n d(y_n, x_n) = \tilde{d}([(y_n)], [(x_n)])$.
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$$\begin{aligned} \tilde{d}([(x_n)], [(z_n)]) &= \lim_n d(x_n, z_n) \\ &\leq \lim_n d(x_n, y_n) + \lim_n d(y_n, z_n) \\ &= \tilde{d}([(x_n)], [(y_n)]) + \tilde{d}([(y_n)], [(z_n)]) \end{aligned}$$

Claim 3: By the definition of $Q(x)$, we have $Q(x) = (x_n) = (x, x, \dots)$ and $Q(y) = (y_n) = (y, y, \dots)$. Then

$$d(Q(x), Q(y)) = \tilde{d}([(x_n)], [(y_n)]) = d(x, y).$$

Claim 4: To show $\overline{Q(X)} = \tilde{X}$ is equivalent to show $Q(X)$ is dense in \tilde{X} . For any $[(x_n)] \in \tilde{X}$, where (x_n) is a Cauchy sequence in X , we show that $[(x_n)]$ can be approximated by elements of $Q(X)$.

Since (x_n) is a Cauchy sequence, for any $\epsilon > 0$, there exists N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$. Take $x = x_N \in X$. Then

$$\tilde{d}([(x_n)], Q(x_N)) = \lim_{n \rightarrow \infty} d(x_n, x_N) \leq \epsilon.$$

Thus, $Q(X)$ is dense in \tilde{X} .

Claim 5:

Let (ξ_k) be a Cauchy sequence in \tilde{X} . For any $\epsilon > 0$, there exists M_1 such that $k, j > M_1$,

$$\tilde{d}(\xi_k, \xi_j) < \frac{\epsilon}{3}.$$

By Claim 4 (density of $Q(X)$), for each i , there exists $z_i \in X$ such that $\tilde{d}(Q(z_i), \xi_i) < \frac{1}{i}$.

Let $\eta_k := Q(z_k)$. There exists $M_2 > M_1$, such that $k, j > M_2$ and $1/k < \epsilon/3, 1/j < \epsilon/3$, then

$$\tilde{d}(\eta_k, \eta_j) \leq \tilde{d}(\eta_k, \xi_k) + \tilde{d}(\xi_k, \xi_j) + \tilde{d}(\eta_j, \xi_j) < \epsilon$$

It follows that (η_k) is a Cauchy sequence in \tilde{X} . Since Q is an isometry, (z_k) is Cauchy in X : $d(z_k, z_j) = \tilde{d}(\eta_k, \eta_j)$.

Let $\zeta := [(z_n)] \in \tilde{X}$. It remains to show $\xi_i \rightarrow \zeta$ in \tilde{X} , i.e., $\lim_{i \rightarrow \infty} \tilde{d}(\xi_i, \zeta) = 0$.

Let $\epsilon > 0$. Choose N_1 such that $i, j \geq N_1$ implies $\tilde{d}(\xi_i, \xi_j) < \epsilon/3$. Choose $N_2 \geq N_1$ such that $i \geq N_2$ implies $1/i < \epsilon/3$. Choose $N_3 \geq N_2$ such that $i, j \geq N_3$ implies $\tilde{d}(\eta_i, \eta_j) < \epsilon/3$.

For $i \geq N_3$,

$$\tilde{d}(\xi_i, \zeta) \leq \tilde{d}(\xi_i, \eta_i) + \tilde{d}(\eta_i, \zeta) < \frac{1}{i} + \tilde{d}(Q(z_i), [(z_n)]) < \frac{\epsilon}{3} + d(z_i, z_n) = \frac{\epsilon}{3} + \lim_{k \rightarrow \infty} d(z_i, z_k).$$

Since (z_n) is Cauchy, choose $N_4 \geq N_3$ such that $d(z_l, z_k) < \epsilon/3$ for all $l, k \geq N_4$. Then

$$\lim_{k \rightarrow \infty} d(z_i, z_k) \leq d(z_i, z_{N_4}) + \sup_{k \geq N_4} d(z_{N_4}, z_k) < d(z_i, z_{N_4}) + \frac{\epsilon}{3}.$$

Since $i, N_4 \geq N_3$, $d(z_i, z_{N_4}) < \epsilon/3$. Thus,

$$\lim_{k \rightarrow \infty} d(z_i, z_k) < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3},$$

and

$$\tilde{d}(\xi_i, \zeta) \leq \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon$$

Hence, $\xi_i \rightarrow \zeta$ in \tilde{X} , so (X_e, d_e) is complete.